BIRTH OF LIMIT CYCLES FROM A 3D TRIANGULAR CENTER OF A PIECEWISE SMOOTH VECTOR FIELD

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ABSTRACT. We consider a piecewise smooth vector field in $\mathbb{R}^3$ where the switching set is on an algebraic variety expressed as the zero of a Morse function. We depart from a model described by piecewise constant vector fields with a non usual center that is constant on the sliding region. Given a positive integer k, we produce suitable non linear small perturbations of the previous model and we obtain piecewise smooth vector fields having exactly k hyperbolic limit cycles instead of the center. Moreover, we also obtain suitable non linear small perturbations of the first model and piecewise smooth vector fields having a unique limit cycle of multiplicity k instead of the center. As consequence, the initial model has codimension infinity. Some aspects of asymptotical stability of such system are also addressed in this paper.

1. INTRODUCTION

1.1. **Setting the problem.** Going a step beyond the classical theory of topological dynamical systems, the theory of the so called *piecewise smooth vector fields* (PSVFs, for short) has been used to model a large number of phenomena. Indeed, both applications and theory of such a brand new area has been widely explored recently. Applications of piecewise smooth vector fields include but not limited to physics, control systems, electrical engineering and problems involving impact, among others. Some landmarks we can quote of such a theory are the work of Teixeira concerning manifolds with boundary (see [18]) and the book of Filippov, see [10]. One can also see the works [5, 6, 7, 12, 13, 14, 19, 20].

Basically, the difference between the classical theory concerning smooth dynamics and our approach lies in the fact that the late assume the existence of a codimension one variety Σ separating the phase portrait in two or more regions, being defined in each region a distinct vector field.

In [10] Filippov provided a convention to induce a vector field even on Σ, which is usually called *Filippov (or sliding) vector field*. Generally, the first papers addressing PSVFs dealt with a variety Σ generated by the preimage of a regular value of some smooth function. Meanwhile such a theory is enhanced by a huge range of applications, even when the discontinuity set is an algebraic variety. Models of PSVFs having Σ in such a shape can be found in books of control or in papers taking into account relay systems (see, for instance, the book of Barbashin - [3]).

In this paper we analyse some qualitative aspects of a PSVF in $\mathbb{R}^3$ whose flow becomes confined on the *switching manifold* Σ after a finite time. Indeed, while in dimension two the dynamics on Σ does not present any tricky behavior, in dimension 3 one may observe some very interesting phenomena. In the present work, the switching manifold is formed by

twelve parts (semi-planes), which give rise to eight different vector fields, defined in eight octants of $\mathbb{R}^3$.

In this scenario, we depart from a piecewise constant model with a non usual center (see Figure 1) that is constant on the sliding regions. So, we perform perturbations in order to study the existence of minimal sets as well as aspects concerning asymptotical and structural stability. Related papers that treat about the asymptotical stability in the context of PSVF are [11] and [19], for example.
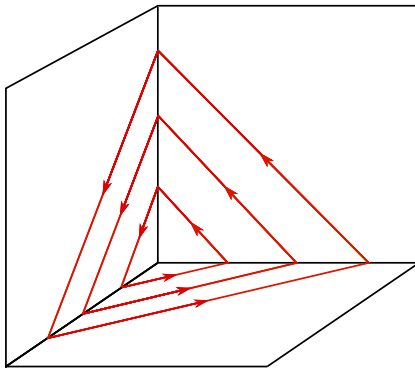

FIGURE 1. The center contained in the switching manifold

We should stress that while we present an abstract model in order to study some aspects of PSVFs having a tricky switching manifold, the model we present is inspired in a formalism that is known in the literature as *complementarity problem* (CP) (see [17] and references therein), which has a strongly connection with PSVFs and its importance in applications has been largely explored. More precisely, our approach in this work has some resemblance with the setting presented in [4] and briefly discussed in [17] (see pages 161–166). Observe that complementarity problems as DVIs (differential variational inequalities) or DCIs (differential complementarity problem) seems to be very important in describing physical situations presenting mechanical impacts or Coulomb friction. The connection between both, piecewise smooth's formalism and complementarity has been reported in the books of Acary and Brogliato [1] (Chapters 1 and 2), di Bernardo et al [7] (Section 2.3.1) and Stewart [17] (Chapter 1).

1.2. **Statements of the main results.** Consider the PSVF Z_0 defined in $\mathbb{R}^3$ by

$$(1) \qquad Z_0(x, y, z) = X_i(x, y, z), \quad \text{if } (x, y, z) \in R_i,\ i = 1, 2, \ldots, 8,$$

where each region R_i, $i = 1, 2, \ldots, 8$ is determined according to the following table:

	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8
sign of x	≥ 0	≤ 0	≤ 0	≥ 0	≥ 0	≤ 0	≤ 0	≥ 0
sign of y	≥ 0	≥ 0	≤ 0	≤ 0	≥ 0	≥ 0	≤ 0	≤ 0
sign of z	≥ 0	≥ 0	≥ 0	≥ 0	≤ 0	≤ 0	≤ 0	≤ 0

By considering a coherent arrangement of the regions listed previously, we get the switching region $\Sigma = \bigcup_{\{j,k\} \in I} \Sigma_{j,k}$, where $\Sigma_{j,k}$ is the boundary separating two adjacent regions R_j and R_k, for $j, k \in I = \{1, 2, \ldots, 8\}$. We call Ω^r the space of vector fields $Z : \mathbb{R}^3 \to \mathbb{R}^3$ having the form of Equation (1) and observe that on Σ we have defined two or more vector

fields, in this case the classical Filippov's convention does not apply. Two approaches taking into account piecewise smooth systems having more than two vector fields defined on can be found in [9] and [15].

In our model, we consider the following choice of constant (linear) vector fields X_i, $i = 1, 2, \ldots, 8$:

$$
\begin{aligned}
X_1(x, y, z) &= (-1, -1, -1), \\
X_2(x, y, z) &= (1, -1, 3), \\
X_3(x, y, z) &= (1, -1, 1), \\
X_4(x, y, z) &= (3, 1, -1), \\
X_5(x, y, z) &= (-1, 3, 1), \\
X_6(x, y, z) &= (-1, 1, 1), \\
X_7(x, y, z) &= (1, 1, 1), \\
X_8(x, y, z) &= (1, 1, -1).
\end{aligned}
$$

(2)

In what follows, we write

$$
S = \Sigma_{1,2} \cup \Sigma_{1,4} \cup \Sigma_{1,5}.
$$

Latter on, we show that S is the sliding region of Σ and it is a global attractor of System (1) (see Lemmas A and B, respectively).

Now we state the main results of the paper.

Theorem A. *Let $Z_0 \in \Omega^r$ be given by (1). For each integer $k \geq 0$, there exists an one-parameter family $Z_\varepsilon \in \Omega^r$ satisfying:*

(a) *$Z_\varepsilon \to Z_0$ when $\varepsilon \to 0$;*
(b) *Restricted to S, Z_ε has exactly k hyperbolic limit cycles in a neighborhood of the origin. The same holds for $k = \infty$ and,*
(c) *Z_0 possesses the invariant subset $S \subset \Sigma$ as a global attractor.*

In order to announce the next result, let us roughly define the classical notion of codimension of vector fields.

Definition 1. *Consider $\Theta(W)$ the set of all small perturbations of a vector field W, defined on a compact set K. We say that W has codimension k if it appears exactly k distinct topological types of vector fields in $\Theta(W)$.*

An immediate consequence of Theorem A is:

Corollary B. *The PSVF $Z_0 \in \Omega^r$, given by (1), has infinite codimension.*

Moreover, the next theorem brightens the scenario around Z_0.

In the next result, consider $\Omega^r(K)$ the space of the PSVF, with domain restricted to a compact subset $K \subset \mathbb{R}^3$ around the origin.

Theorem C. *Let $Z_0 \in \Omega^r(K)$ be given by (1). For each neighborhood $\mathcal{V}$ of Z_0 and for each integer $k > 0$ there exists $Z_\varepsilon^k \in \mathcal{V}$ having codimension k.*

Next result also provides informations about the asymptotical stability of the origin.

Theorem D. *There exist $Z_\varepsilon^0 \in \Omega^r$, an one-parameter family of perturbations of the PSVF Z_0, for which the origin is either asymptotically stable or unstable, depending on the sign of ε.*

Theorems A and C and Corollary B are proved in Section 5. Theorem D is proved in Section 3.

The paper is organized as follows. In Section 2 we introduce the terminology, some definitions and the basic theory about PSVFs. In Section 3 we study the main properties of System Z_0. In Section 4 suitable perturbations of this system are considered. Moreover, the birth of limit cycles are explicitly exhibited. In Section 5 we prove the main results of the paper. Finally, in Section 6 we present a brief conclusion.

2. Preliminaries

Inspired in the models presented in [4] (see also [17]) we first consider the particular PSVF Z_0 given by (1). At the sequel, we formalize some theory about this kind of vector field.

Denote $\pi_i : \mathbb{R}^3 \longrightarrow \mathbb{R}$ the projection onto the $i-$th coordinate, $i = 1, 2, 3$ and observe that

$$\Sigma = \bigcup_{i=1,2,3} \pi_i^{-1}(0).$$

We remark that once Σ is not regular, the coordinate planes $A_i = \pi_i^{-1}(0)$ itself, $i = 1, 2, 3$, do not play any special role in our approach. However, they allows us to define the function $h : \mathbb{R}^3 \setminus \mathcal{F}_1 \longrightarrow \mathbb{R}$ given by

$$h = \chi_{A_1} \pi_1 + \chi_{A_2} \pi_2 + \chi_{A_3} \pi_3,$$

where χ_{A_i} is the characteristic function associated to the set A_i, $i = 1, 2, 3$, and $\mathcal{F}_1$ is given by

$$\mathcal{F}_1 = \bigcup_{i,j \in \{1,2,3\}} \left[\pi_i^{-1}(0) \cap \pi_j^{-1}(0) \right].$$

Note that the set of three coordinate axes $\mathcal{F}_1$ is not a codimension one variety of $\mathbb{R}^3$. So, the Filippov convention does not apply on $\mathcal{F}_1 \subset \Sigma$ directly.

Moreover, let $X_k : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be C^r-vector fields, $k = 1, \ldots, 8$ and designate by $\mathfrak{X}^r$ the space of C^r-vector fields on $\mathbb{R}^3$ endowed with the C^r-topology with $r = \infty$ or $r \geq 1$ large enough for our purposes.

In order to classify the points on Σ, for each vector field X_i, $i = 1, \ldots, 8$, we denote $X_i h(p) = \langle X_i(p), \nabla h(p) \rangle$ and $(X_i^n h(p) = \langle X_i(p), \nabla X_i^{n-1} h(p) \rangle)$, the Lie's derivatives, where $\langle \cdot, \cdot \rangle$ denote the canonical inner product. We may consider Ω^r endowed with the product topology and denote any element in Ω^r by $Z = (X_1, \ldots, X_8)$, which we will accept to be multivalued in points of Σ. The basic results of differential equations, in this context, were stated by Filippov [10]. Related theories can be found in [7, 16, 20] and references therein.

On $\Sigma_{ij} \setminus \mathcal{F}_1$ we generically distinguish three regions according to the Filippov's convention (see [10]):
- Sewing Region: $\Sigma_{ij}^c = \{p \in \Sigma_{ij} \mid X_i h(p) X_j h(p) > 0\}$.
- Sliding Region: $\Sigma_{ij}^s = \{p \in \Sigma_{ij} \mid X_i h(p) < 0 \text{ and } X_j h(p) > 0\}$.
- Escaping Region: $\Sigma_{ij}^e = \{p \in \Sigma_{ij} \mid X_i h(p) > 0 \text{ and } X_j h(p) < 0\}$.

We stress that for those points $q \in \Sigma_{ij}^s \cup \Sigma_{ij}^e$, it was established by Filippov (see [10]) a manner to induce a vector field Z^s on Σ_{ij} which is tangent to $\Sigma_{ij}^s \cup \Sigma_{ij}^e$ and writes

$$(3) \qquad Z_{ij}^s = \frac{1}{\langle (X_j - X_i), \nabla h \rangle} \left[\langle X_j, \nabla h \rangle X_i - \langle X_i, \nabla h \rangle X_j \right] = \frac{X_j h X_i - X_i h X_j}{X_j h - X_i h}.$$

In this work, the vector field Z_{ij}^s will be called **sliding vector field** or **Filippov vector field**.

By a time rescaling, the sliding vector field Z_{ij}^s is orbitally equivalent on Σ_{ij}^s to the **normalized sliding vector field**

$$\widetilde{Z}_{ij}^s = [\langle X_j, \nabla h \rangle X_i - \langle X_i, \nabla h \rangle X_j] = X_j h X_i - X_i h X_j.$$

Moreover, since the denominator in (3) could be negative on Σ_{ij}^e, we get that $\widetilde{Z}_{ij}^s$ is orbitally equivalent to $-Z_{ij}^s$ on Σ_{ij}^e.

Note that $\widetilde{Z}_{ij}^s$ can be C^r extended beyond the boundary of $\Sigma_{ij}^s \cup \Sigma_{ij}^e$.

In this paper we deal with the following notion of flow.

Definition 2. *The flow of $Z = (X_1, \ldots, X_8) \in \Omega^r$ is obtained by the concatenation of flows of $X_1, \ldots, X_8$ or X_{ij}^s, $i, j \in \{1, \ldots, 8\}$.*

Let p be a sewing point contained in Σ_{j_1, k_1}. In our approach, we allow the trajectory of X_i by $p \in \Sigma_{j_1, k_1}$ to hit Σ_{j_2, k_2} with $\{j_1, k_1\} \neq \{j_2, k_2\}$ after a finite time. Let $t_1(p) > 0$ be the X_i-*flight time* employed in such journey. So we define the *return map associated to X_i* by $\varphi_{Z_i}(p) = \phi_{Z_i}(t_1(p), p) = p_1 \in \Sigma_{j_2, k_2}$. When it is possible, that is, p_1 is still a sewing point, we repeat this procedure for it. Assuming that the trajectory which started in p_0 hits only sewing point, the *complete return map* associated to Z_0, given by equation (1), is defined by the composition of these return maps, i.e.,

$$\varphi_{Z_0}(p) = \varphi_{Z_8} \circ \ldots \circ \varphi_{Z_1}(p).$$

The starting vector field X_1 can be changed by any other X_j, $j = 2, \ldots, 8$ since the orientation is preserved.

Alternatively, inspired in the previous ideas, if the starting point p belongs to a sliding or escaping region, one can still define a complete return map. Indeed, in the next section we construct such a map for the region S, since the results throughout this paper address this region. The construction of such maps provide an important object in order to study the behavior of Z_0 around the origin. The proofs of the main results require a detailed analysis of the complete return map.

In this paper, we study smooth nonlinear perturbations of the model (1) and a complete picture of its dynamics is exhibited. It is worth to say that some constructions and ideas of [8] are very useful in our approach.

3. Auxiliary results

In this section we describe some important features about the PSVF expressed by Equation (1). Indeed, in Subsection 3.1 we describe its behavior in Σ and the sliding vector field associated to Equation (1) (Lemmas A and B). Then, we study the sliding region S of Σ and the asymptotical stability at origin of an one-parameter family of perturbation of Z_0. Later on we analyse the way in which the trajectories converge to a limit set. This last analysis permits us to detect a global attractor for the trajectories of Equation (1).

3.1. **Properties of System Z_0 given by Equation** (1)**.** In this Subsection we analyze the region of discontinuity Σ. Indeed, the behavior of Z_0 in each component of Σ is presented by the following two lemmas.

Lemma A. The set $\Sigma \setminus S$ is contained in Σ^c.

Proof. It is not difficult to verify that $\Sigma \setminus S$ coincides with the sewing region. For instance, if $p = (x, 0, z) \in \Sigma_{2,3}$ then $h(p) = \pi_2(p)$ and consequently

$$X_2 \pi_2(p) \cdot X_3 \pi_2(p) = \langle (1, -1, 3), (0, 1, 0) \rangle \cdot \langle (1, -1, 1), (0, 1, 0) \rangle = 1 > 0.$$

The other cases are analogous. □

Lemma B. The set $S = \Sigma_{1,2} \cup \Sigma_{1,4} \cup \Sigma_{1,5}$ is the sliding region Σ^s and the respective sliding vectors fields Z_{12}^s, Z_{14}^s and Z_{15}^s are given by

$$
\begin{aligned}
Z_{12}^s(x, y, z) &= (0, -1, 1), \\
Z_{14}^s(x, y, z) &= (1, 0, -1), \\
Z_{15}^s(x, y, z) &= (-1, 1, 0).
\end{aligned}
$$

Proof. The proof of Lemma B is straightforward following Lemma A and using an algebraic manipulation of the formula (3). □

Now we define a complete return map associated to the sliding vector fields of Z_0 given by (1). Indeed, consider the regions

$$
\begin{aligned}
S_1 &= \Sigma_{1,2} \cap \Sigma_{1,5}, \\
S_2 &= \Sigma_{1,2} \cap \Sigma_{1,4}, \\
S_3 &= \Sigma_{1,4} \cap \Sigma_{1,5},
\end{aligned}
$$

which are transversal sections of the flows $Z_{12}^s, Z_{14}^s, Z_{15}^s$.

Given a point $s_1 \in S_1$, consider the transition map $P_1 : S_1 \to S_2$ such that $P_1(s_1) = s_2$ is the point of S_2 where the Z_{12}^s-trajectory passing through s_1 intersects S_2. Analogously, define the transition maps $P_2 : S_2 \to S_3$ and $P_3 : S_3 \to S_1$ using the flows of Z_{14}^s and Z_{15}^s respectively.

Consequently the complete return map for the sliding region writes

$$\varphi_{Z_0} = P_3 \circ P_2 \circ P_1 : S_1 \to S_1.$$

Remark 1. *In our particular case, the forward trajectory (which is the interesting situation here) has a unique choice, that is, the forward trajectory of a point in $\mathcal{F}_1$ is well defined and it does not depart from the set S.*

The following Propositions A and B describe the behavior on S.

Proposition A. The set S is fulfilled with periodic trajectories of (1).

Proof. We make use of the first return map in order to guarantee that any trajectory through a point in $S = \Sigma_{1,2} \cup \Sigma_{1,4} \cup \Sigma_{1,5}$ is closed. Indeed, consider S_1 the region of return of the trajectories and an initial condition $(0, y_0, 0) \in S_1$.

By Lemma B, the flows $\varphi_t^{1,2}$, $\varphi_t^{1,4}$ and $\varphi_t^{1,5}$ starting in $p_{1,2}^0 = (0, y_0, z_0), p_{1,4}^0 = (x_0, 0, z_0)$ and $p_{1,5}^0 = (x_0, y_0, 0)$ associated to the sliding vector fields Z_{12}^s, Z_{14}^s and Z_{15}^s, respectively, are

$$\begin{array}{rcl} \varphi_t^{1,2}(p_{12}^0) & = & (0, -t + y_0, t + z_0), \\ \varphi_t^{1,4}(p_{14}^0) & = & (t + x_0, 0, -t + z_0), \\ \varphi_t^{1,5}(p_{15}^0) & = & (-t + x_0, t + y_0, 0). \end{array}$$

(4)

Now, once $S_1 = \Sigma_{1,2} \cap \Sigma_{1,5}$ and every future trajectory reaches S_1 transversally for positive time, the trajectory enters in $\Sigma_{1,2}$ and consequently is governed by the flow $\varphi_t^{1,2}$. Nevertheless, considering $p_0 = (0, y_0, 0) \in S_1$ we obtain $\varphi_t^{1,2}(p_0) = (0, -t + y_0, t)$ and consequently $P_1(p_0) = \varphi_{t=y_0}^{1,2}(p_0) = (0, 0, y_0) \in S_2$.

From the point $(0, 0, y_0)$ the trajectory enters in $\Sigma_{1,4}$ through the flow $\varphi_t^{1,4}$ and we get $\varphi_t^{1,4}(0, 0, y_0) = (t, 0, -t + y_0)$. Again, taking $t = y_0$, we have $P_2(0, 0, y_0) = \varphi_{t=y_0}^{1,4}(0, 0, y_0) = (y_0, 0, 0) \in S_3$. Finally, the flow $\varphi_t^{1,5}$ starting at the point $(y_0, 0, 0)$ is $\varphi_t^{1,5}(y_0, 0, 0) = (-t + y_0, t, 0)$, then $P_3(y_0, 0, 0) = \varphi_{t=y_0}^{1,5}(y_0, 0, 0) = (0, y_0, 0) = p_0 \in S_1$, that is, the first return map $\varphi_{Z_0} : S_1 \to S_1$ is given by

$$\varphi_{Z_0}(y_0) = \varphi_{t=y_0}^{1,5} \circ \varphi_{t=y_0}^{1,4} \circ \varphi_{t=y_0}^{1,2}(y_0) = y_0.$$

Therefore through every point $p_0 = (0, y_0, 0) \in S_1$ passes a periodic trajectory, which means that $S = \Sigma_{1,2} \cup \Sigma_{1,4} \cup \Sigma_{1,5}$ is fulfilled with periodic orbits. □

Figures 2 and 3 illustrate the situation described in Proposition A. As we can see, the origin of the system of coordinates looks like a center once every neighborhood of it in S is fulfilled with periodic trajectories. Moreover, Proposition B tells that this continuum of periodic trajectories is a global attractor of System (1).
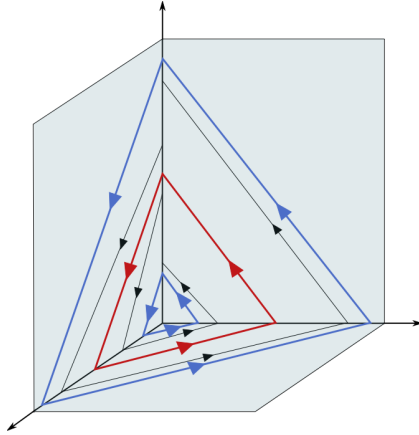

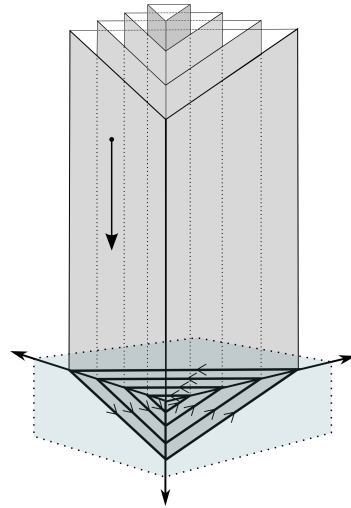FIGURE 2. S is fulfilled by periodic trajectories.

FIGURE 3. The trajectories going to the 3D triangular center.

The next proposition tell us that every trajectory of Z_0 in $\mathbb{R}^3$ converges to S.

Proposition B. The set S is a global attractor for Z_0.

Proof. We start considering the trivial case, that is, taking a point $q \in S = \Sigma_{1,2} \cup \Sigma_{1,4} \cup \Sigma_{1,5}$. Then, by Proposition A, the Z_0-trajectory through q is periodic and we get the result in this case. Now, consider a point $p_1 = (x_1, y_1, z_1) \in R_1$ and $\varphi_t^1(p_1)$ the flow associated to the vector field Z_1. Clearly $\varphi_t^1(p_1) = (-t + x_1, -t + y_1, -t + z_1)$. Thus, taking $t_1 = \min\{x_1, y_1, z_1\}$ it holds $\varphi_{t_1}^1(p_1) = q \in S$ which is periodic, so the result is true.

Now we consider a point p in $\bigcup_{i=2,\ldots,8} R_i$ and we must prove that p converges to S. Indeed, without loss of generality, take $p = p_6 \in \Sigma_{5,6} \setminus S$ since all points in R_5 reaches $\Sigma_{5,6}$. Observe that the trajectory φ_t^6 through $p_6 = (x_6, y_6, z_6)$ writes $\varphi_t^6(p_6) = (-t + x_6, t + y_6, t + z_6)$. Once $y_6 > 0$ and $t + y_6$ is increasing in t we get $\varphi_t^6(p_6) \cap \Sigma_{6,7} = \emptyset$. Consequently, each point starting in $\Sigma_{5,6}$ reaches $\Sigma_{2,6}$ in a finite time in sewings points and enters in the region R_2 where the dynamics is governed by the flow associated to this region, namely, φ_t^2.

On the region R_2, the flow starting on a point $p_2 = (x_2, y_2, 0) \in \Sigma_{2,6}$ is given by $\varphi_t^2(p_2) = (t + x_2, -t + y_2, 3t)$, where $x_2 < 0$ and $y_2 > 0$. Note that the trajectory through p_2 meets $\Sigma_{1,2}$ when $t + x_2 = 0$, that is, for $t = -x_2 > 0$. On the other hand, if $t = y_2$, then $\varphi_{t=y_2}^2(p_2) \in \Sigma_{2,3}$. Consequently, the plane $\Gamma_2 = \{(x, y, z) \in \mathbb{R}^3 \mid y = -x, x < 0, y > 0\}$ separates the region R_2 into two disjoint parts, one of them having the set S as it ω-limit and the other one reaching the set $\Sigma_{2,3}$.

Those points which reach $\Sigma_{2,3}$ have a behavior similar to the points of $\Sigma_{5,6}$. In fact, all they go to $\Sigma_{3,4}$ and the plane $\Gamma_4 = \{(x, y, z) \in \mathbb{R}^3 \mid z = -y, y < 0, z > 0\}$ separates the region R_4 into two disjoint parts, one of them having the set S as it ω-limit and the other one reaching the set $\Sigma_{4,8}$.

Again, those points which reach $\Sigma_{4,8}$ has a behavior similar to the points of $\Sigma_{5,6}$ and $\Sigma_{2,3}$. In fact, all they go to $\Sigma_{5,8}$ and the plane $\Gamma_5 = \{(x, y, z) \in \mathbb{R}^3 \mid z = -x, z < 0, x > 0\}$ separates the region R_5 into two disjoint parts, one of them having the set S as it ω-limit and the other one reaching the set $\Sigma_{5,6}$.

Observe that each one of the planes Γ_2, Γ_4 and Γ_5 guides half of the points in $\Sigma_{2,6}$, $\Sigma_{3,4}$ and $\Sigma_{5,8}$, respectively, until S. In this way from the domain $\Sigma_{5,6}$ of the complete first return map, just the points of the region $\Phi_1 = \{(0, y, z) \in \Sigma_{5,6} \mid 0 < y < -z/8\}$ return to $\Sigma_{5,6}$. In fact, saturating by the flow the straight line $r_1 = \{(0, y, z) \in \Sigma_{5,6} \mid y = -z/8\}$ we obtain that it goes to the straight line $r_2 = \Sigma_{5,8} \cap \Gamma_5$.

Repeating the previous argument, we get that the complete first return map φ_{Z_0} satisfies $\varphi_{Z_0}^2(\Phi_1 \setminus \Phi_2) \to S$, where $\Phi_2 = \{(0, y, z) \in \Sigma_{5,6} \mid 0 < y < -z/64\}$. Recursively, $\varphi_{Z_0}^n(\Phi_{n-1} \setminus \Phi_n) \to S$, where $\Phi_n = \{(0, y, z) \in \Sigma_{5,6} \mid 0 < y < -z/8^n\}$. Therefore, the result follows. $\qquad\square$

Corollary 1. *Given a point in $q \in \mathbb{R}^3$ it is possible to determine the number of times that the trajectory through q hits a connect component of Σ before reaches S.*

Proof. We write the proof for one connect component of Σ, for example $\Sigma_{5,6}$. The proof for the others components is similar. Given a point $q \in \mathbb{R}^3$, let $q_1 = (0, y_1, z_1)$ the intersection of its trajectory with $\Sigma_{5,6}$. Following the final steps of the proof of the previous proposition we can see that if $|y_1|/|z_1| > 1/8$ then the trajectory by q converges to S before it returns to $\Sigma_{5,6}$. Repeating the same argument, if $1/8 > |y_1|/|z_1| > 1/64$ then the trajectory through q converges to S after hits $\Sigma_{5,6}$ once.

In general, if $1/8^{n-1} > |y_1|/|z_1| > 1/8^n$ then the trajectory through q converges to S before it hits $\Sigma_{5,6}$ a number $n - 1$ of times. $\qquad\square$

3.2. **Asymptotic stability at the origin of a perturbation of Z_0.** In this subsection we exhibit a perturbation Z_ε^0 of the system Z_0 for which the behavior of the trajectories of the model presented in [4] can be topologically reproduced. We stress that the model introduced in [4] does not present asymptotical stability at the origin once the trajectories scape from the origin and goes to infinity (which happens in case $\varepsilon < 0$ in Lemma 1). Besides, if $\varepsilon > 0$ in Lemma 1, we prove that the Z_ε^0 is asymptotically stable at origin.

Consider initially the smooth function $F_\varepsilon^0 : \mathbb{R} \to \mathbb{R}$, such that

$$F_\varepsilon^0(x) = \varepsilon x.$$

Let be Z_0 given by (1) and

$$(5) \qquad Z_\varepsilon^0(x, y, z) = X_{i,\varepsilon}^0(x, y, z), \quad \text{if } (x, y, z) \in R_i, \ i = 1, 2, \ldots, 8,$$

where

$$(6) \qquad X_{2,\varepsilon}^0 \ = X_2 + \left(0, 2\frac{\partial F_\varepsilon^0(z)}{\partial z}, 0\right),$$

and $X_{i,\varepsilon}^0(x, y, z) = X_i(x, y, z)$ for $i = 1, 3, 4, 5, 6, 7, 8$ are given by (2). Associated to the perturbation (6) we have the normalized sliding vector fields given by

$$Z_{1,4,\varepsilon}^{0,s}(x, y, z) \ = Z_{1,4}^s(x, y, z) = (1, 0, -1)$$

$$(7) \qquad Z_{1,5,\varepsilon}^{0,s}(x, y, z) \ = Z_{1,5}^s(x, y, z) = (-1, 1, 0)$$

$$Z_{1,2,\varepsilon}^{0,s}(x, y, z) \ = \left(0, -1 + \frac{\partial F_\varepsilon^0(z)}{\partial z}, 1\right).$$

So, we get that dynamics is governed by the flow $\phi_{Z_{1,2,\varepsilon}^{0,s}}$ of $Z_{1,2,\varepsilon}^0$ given in (7) is parameterized by

$$(8) \qquad \phi_{Z_{2,\varepsilon}^{0,s}}(t, p) \ = (x, -t + F_\varepsilon^0(t) + y, t + z),$$

where $p = (x, y, z)$.

In the following, we present the result that treat of the stability of Z_ε^0:

Lemma 1. *Consider the perturbation Z_ε^0, given in (5) of Z_0. We get*

(a) *if $\varepsilon > 0$ then the origin is asymptotically stable;*
(b) *if $\varepsilon < 0$ then the origin is asymptotically unstable.*

Proof. First of all, we prove that S is invariant and a global attractor for Z_ε^0. In fact, we have that the sliding vector fields $Z_{1,4,\varepsilon}^{0,s}$ and $Z_{1,5,\varepsilon}^{0,s}$ coincides with the unperturbed case. Given $p \in \Sigma_{12}$, by (8) we get that trajectories of Z_ε^0 intercepts the axis z (frontier of Σ_{12}) and then follows the trajectory of $Z_{1,4,\varepsilon}^{0,s}$, that coincides with $Z_{1,4}^{0,s}$. Therefore, we conclude that S is invariant for Z_ε^0. The proof that S is a global attractor of Z_ε^0 is similar to the proof of Proposition B and will be omitted.

As S is invariant and a global attractor of Z_ε^0, to characterize the stability is necessary and sufficient to study the stability of the first return map $\varphi_{Z_\varepsilon^0} = \phi_{Z_{1,2,\varepsilon}^{0,s}} \circ \phi_{Z_{1,5,\varepsilon}^{0,s}} \circ \phi_{Z_{1,4,\varepsilon}^{0,s}}$.

We consider the transverse section on S, the domain of $\varphi_{Z_\varepsilon^0}$, given by $S_2 = \{(0, 0, z); z > 0\}$.

So, we have $\phi_{Z_{1,2,\varepsilon}^{0,s}} \circ \phi_{Z_{1,5,\varepsilon}^{0,s}} \circ \phi_{Z_{1,4,\varepsilon}^{0,s}}(t, (0, 0, z_0)) = \phi_{Z_{1,2,\varepsilon}^{0,s}}(t, (0, z_0, 0))$. The first return map is given by the third coordinate of the point in S_2 provided by the intersection of the

graph of function $G^0(z) = -z + F_\varepsilon^0(z) + z_0$ with the half straight line S_2. Therefore, $\varphi_{Z_\varepsilon^0}$ satisfies

$$z - \varphi_{Z_\varepsilon^0}(z) - F_\varepsilon^0(\varphi_{Z_\varepsilon^0}(z)) = 0.$$

Explicitly, we get that $\varphi_{Z_\varepsilon^0}(0,0,z) = \left(0, 0, \dfrac{z}{1+\varepsilon}\right)$. Therefore, if $\varepsilon > 0$ then $\lim\limits_{n\to\infty}(\varphi_{Z_\varepsilon^0}^n(0,0,z)) = (0,0,0)$ and if $\varepsilon < 0$ then $\lim\limits_{n\to-\infty}(\varphi_{Z_\varepsilon^0}^n(0,0,z)) = (0,0,0)$. $\hfill\square$

Proof of Theorem D. The proof is straightforward using Lemma 1. $\hfill\square$

4. Properties of a specific perturbation of System Z_0

4.1. **Analysis of the first return map of the perturbed vector field.** Consider the C^∞-functions $F_\varepsilon^i(x) : \mathbb{R} \to \mathbb{R}$, where $i = 1, 2, 3$, such that

$$(9) \qquad\qquad F_\varepsilon^1(x) \;=\; \varepsilon x(\varepsilon - x)(2\varepsilon - x)\dots(k\varepsilon - x),$$

$$(10) \qquad\qquad F_\varepsilon^2(x) \;=\; x\sin(\pi\varepsilon^2/x),$$

$$(11) \qquad\qquad F_\varepsilon^3(x) \;=\; \varepsilon(x - \varepsilon)^{k+1},$$

with $k \in \mathbb{N}$.

Lemma 2. *Consider the function $F_\varepsilon^1(x)$ given by (9).*
- *(a) If $\varepsilon < 0$ then F_ε^1 does not have roots in $(0, +\infty)$.*
- *(b) If $\varepsilon > 0$ then F_ε^1 has exactly k roots in $(0, +\infty)$ and these roots are $\{\varepsilon, 2\varepsilon, \dots, k\varepsilon\}$.*
- *(c) $\dfrac{\partial F_\varepsilon^1}{\partial x}(j\varepsilon) = \varepsilon^{k+1}(-1)^{j-1}j!(k-j)!$ for $j \in \{1, 2, \dots, k\}$. It means that such partial derivative at $j\varepsilon$ is positive for j odd and negative for j even.*

Proof. When $x > 0$, by a straightforward calculation $F_\varepsilon^1(x) = 0$ if, and only if, $(\varepsilon - x)(2\varepsilon - x)\dots(k\varepsilon - x) = 0$. So, the roots of $F_\varepsilon^1(x)$ in $(0, +\infty)$ are $\varepsilon, 2\varepsilon, \dots, k\varepsilon$. Moreover,

$$\frac{\partial F_\varepsilon^1}{\partial x}(x) = \varepsilon(\varepsilon - x)(2\varepsilon - x)\dots(k\varepsilon - x) - \sum_{i=1, i\neq j}^{k}\varepsilon x(\varepsilon i - x).$$

Therefore,

$$\frac{\partial F_\varepsilon^1}{\partial x}(j\varepsilon) \;=\; \varepsilon(j\varepsilon)(\varepsilon - j\varepsilon)(2\varepsilon - j\varepsilon)\dots((j-1)\varepsilon - j\varepsilon)((j+1)\varepsilon - j\varepsilon)$$

$$((j+1)\varepsilon - j\varepsilon)\dots(k\varepsilon - j\varepsilon)$$

$$= \varepsilon^{k+1}(-1)^{j-1}j!(k-j)!$$

This proves items (b) and (c). Item (a) follows immediately. $\hfill\square$

Lemma 3. *Consider the function $F_\varepsilon^2(x)$ given by expression (10). For $\varepsilon \neq 0$ the function F_ε^2 has infinitely many roots in $(0, \varepsilon^2)$, these roots are $\{\varepsilon^2/2, \varepsilon^2/3, \dots\}$ and*

$$\frac{\partial F_\varepsilon^2}{\partial x}(\varepsilon^2/j) = \mp\pi j \text{ for } j \in \{2, 3, \dots\}.$$

It means that such derivative at (ε^2/j) is positive for j odd and negative for j even.

Proof. When $x > 0$, by a straightforward calculation $F_\varepsilon^i(x) = 0$ if, and only if, $\sin(\pi\varepsilon^2/x) = 0$. So, the roots of $F_\varepsilon^i(x)$ in $(0, \varepsilon^2)$ are $\varepsilon^2/2, \varepsilon^2/3, \ldots$. Moreover,

$$\frac{\partial F_\varepsilon^2}{\partial x}(x) = \sin\left(\frac{\pi\varepsilon^2}{x}\right) - x\cos\left(\frac{\pi\varepsilon^2}{x}\right)\left(\frac{\pi\varepsilon^2}{x^2}\right).$$

Therefore, $\dfrac{\partial F_\varepsilon^2}{\partial x}(\varepsilon^2/j) = \mp\pi j$. $\square$

Lemma 4. *Consider F_ε^3 given in (11). Then $x = \varepsilon$ is a zero of multiplicity k of equation $F_\varepsilon^3(x) = 0$.*

Proof. In fact, by expression (11) we get that $F_\varepsilon^3(\varepsilon) = \dfrac{\partial F_\varepsilon^3}{\partial x}(\varepsilon) = \cdots = \dfrac{\partial^k F_\varepsilon^3}{\partial x^k}(\varepsilon) = 0$ and $\dfrac{\partial^{k+1} F_\varepsilon^3}{\partial x^{k+1}}(\varepsilon) = (k+1)!\varepsilon \neq 0$. $\square$

Consider Z_0 given by (1) and

(12) $$Z_\varepsilon^{1,2,3}(x, y, z) = X_{i,\varepsilon}^{1,2,3}(x, y, z), \quad \text{if } (x, y, z) \in R_i, \ i = 1, 2, \ldots, 8,$$

where

$$X_{2,\varepsilon}^1 = X_2 + \left(0, 2\frac{\partial F_\varepsilon^1}{\partial z}(z), 0\right),$$

(13) $$X_{2,\varepsilon}^2 = X_2 + \left(0, 2\frac{\partial F_\varepsilon^2}{\partial z}(z), 0\right)$$

$$X_{2,\varepsilon}^3 = X_2 + \left(0, 2\frac{\partial F_\varepsilon^3}{\partial z}(z), 0\right)$$

and $X_{i,\varepsilon}^{1,2,3}(x, y, z) = X_i(x, y, z)$ for $i = 1, 3, 4, 5, 6, 7, 8$ are given by (2). Associated to the perturbation (13) we have the normalized sliding vector fields given by

$$Z_{1,4,\varepsilon}^{1,2,3,s}(x, y, z) = Z_{1,4}^s(x, y, z) = (1, 0, -1)$$

$$Z_{1,5,\varepsilon}^{1,2,3,s}(x, y, z) = Z_{1,5}^s(x, y, z) = (-1, 1, 0)$$

(14) $$Z_{1,2,\varepsilon}^{1,s}(x, y, z) = \left(0, -1 + \frac{\partial F_\varepsilon^1}{\partial z}(z), 1\right)$$

$$Z_{1,2,\varepsilon}^{2,s}(x, y, z) = \left(0, -1 + \frac{\partial F_\varepsilon^2}{\partial z}(z), 1\right)$$

$$Z_{1,2,\varepsilon}^{3,s}(x, y, z) = \left(0, -1 + \frac{\partial F_\varepsilon^3}{\partial z}(z), 1\right).$$

A straightforward calculation shows that the trajectories $\phi_{Z_{1,2,\varepsilon}^{i,s}}$ of $Z_{1,2,\varepsilon}^i$, with $i = 1, 2, 3$, given in (14) are parameterized by

$$\phi_{Z_{2,\varepsilon}^{1,s}}(t, p) \quad = (x, -t + F_\varepsilon^1(t) + y, t + z)$$

(15) $$\phi_{Z_{2,\varepsilon}^{2,s}}(t, p) \quad = (x, -t + F_\varepsilon^2(t) + y, t + z)$$

$$\phi_{Z_{2,\varepsilon}^{3,s}}(t, p) \quad = (x, -t + F_\varepsilon^3(t) + y, t + z),$$

where $p = (x, y, z)$.

Remark 2. *Take $Z_\varepsilon = Z_\varepsilon^i$, with $i = 1, 2, 3$. It is easy to see that $Z_\varepsilon \to Z_0$ when $\varepsilon \to 0$, in the C^r-topology.*

The next result takes into account the first return map of the perturbed sliding vector fields. However, from now on we consider the region S_2 as starting point instead of S_1. Of course, in this case the return map can be obtained in a totally analogous way.

Lemma 5. *The first return map of $\varphi_{Z_\varepsilon^{1,2}} : S_2 \to S_2$ satisfies the equation*

$$z - \varphi_{Z_\varepsilon^{1,2}}(z) - F_\varepsilon^{1,2}(\varphi_{Z_\varepsilon^{1,2}}(z)) = 0.$$

Proof. Consider the initial condition $p_0 = (0, 0, z_0) \in S_2$. The first return map is given by $\phi_{Z_{1,2,\varepsilon}^{1,2,s}} \circ \phi_{Z_{1,5,\varepsilon}^{1,2,s}} \circ \phi_{Z_{1,4,\varepsilon}^{1,2,s}}$. By expression (4) and (15) we get that $\phi_{Z_{1,2,\varepsilon}^{1,2,s}} \circ \phi_{Z_{1,5,\varepsilon}^{1,2,s}} \circ \phi_{Z_{1,4,\varepsilon}^{1,2,s}}(t, (0, 0, z_0)) = \phi_{Z_{1,2,\varepsilon}^{1,2,s}}(t, (0, z_0, 0))$. So, as we get in the proof of Proposition 1, the first return map is given by the third coordinate of the point in S_2 given by the intersection of the graph of function $G^{1,2}(z) = -z + F_\varepsilon^{1,2}(z) + z_0$ with the half straight line S_2. Therefore, $\varphi_{Z_\varepsilon^{1,2}}$ satisfies

$$z - \varphi_{Z_\varepsilon^{1,2}}(z) - F_\varepsilon^{1,2}(\varphi_{Z_\varepsilon^{1,2}}(z)) = 0.$$

□

Lemma 6. *The first return map $\varphi_{Z_\varepsilon^3}$ associated to Z_ε^3 is expressed by*

$$\varphi_{Z_\varepsilon^3}(z) = z + \varepsilon(z - \varepsilon)^{k+1}.$$

Proof. As the previous case, we get that $\varphi_{Z_\varepsilon^3}(0, 0, z) = P_1 \circ P_3 \circ P_2(0, 0, z) = P_1(0, z, 0)$ and the application Π_3 is given implicitly as one of the solutions z of

(16) $$G^3(z, \varepsilon) = 0$$

where $G^3(z, \varepsilon) = -z + F_\varepsilon^3(z) + y$. Note that putting $y = \varepsilon$ then $z = \varepsilon$ is a solution of Equation (16). In other words, $P_1(0, \varepsilon, 0) = (0, 0, \varepsilon)$. Considering the change of coordinates $\overline{z} = z - \varepsilon$ and $\overline{y} = y - \varepsilon$ then Equation (16) becomes

$$-\overline{z} + \varepsilon\overline{z}^{k+1} + \overline{y} = 0.$$

Therefore, we obtain

$$\varphi_{Z_\varepsilon^3}(0, 0, z) = z - \varepsilon(z - \varepsilon)^{k+1}.$$

□

4.2. Limit cycles of the perturbed vector fields $Z_\varepsilon^{1,2}$.

Proposition 2. *Consider $Z_\varepsilon^{1,2}$ given by expression (12). Then Z_ε^1 has exactly k limit cycles and Z_ε^2 has infinite many limit cycles, all of them situated in S. Moreover, these limit cycles are hyperbolic repeller limit cycles if j is odd and a hyperbolic attractor limit cycles if j is even.*

Proof. The proof that S is invariant and a global attractor for $Z_\varepsilon^{1,2}$ is analogous that we done in Proposition 1 and we omit it.

So, if there exists limit cycles, then they are situated at S. Moreover when we restrict the flow of $Z_\varepsilon^{1,2}$ to S, by Proposition B, the fixed points of the first return map $\varphi_{Z_\varepsilon^{1,2}} = \phi_{Z_{1,2,\varepsilon}^{1,2,s}} \circ \phi_{Z_{1,5,\varepsilon}^{1,2,s}} \circ \phi_{Z_{1,4,\varepsilon}^{1,2,s}}$ provide us the periodic orbits of systems $Z_\varepsilon^{1,2}$. Since these fixed points are isolated (limit cycles), the stability of each one is given by the stability of the fixed point of the difeomorphism $\varphi_{Z_\varepsilon^{1,2}}$. In both cases, we consider the transverse section on S, the domain of $\varphi_{Z_\varepsilon^{1,2}}$, given by $S_2 = \{(0,0,z); z > 0\}$.

By Lemma 5, we get that the fixed points of the first return maps are given by the zeros of the function $F_\varepsilon^{1,2}(z)$. Considering the first perturbation Z_ε^1, by Lemma 2, we conclude that these roots in S_2 for $\varepsilon > 0$, are given by $\varepsilon, 2\varepsilon, \ldots, k\varepsilon$. Therefore, we get that the fixed points of $\varphi_{Z_\varepsilon^1}$ are given by $p_1 = (0,0,\varepsilon), p_2 = (0,0,2\varepsilon), \ldots, p_k = (0,0,k\varepsilon)$ with $p_i \in S_2, i = 1, \ldots, k$.

So, the stability of these fixed points are given by item (c) of Lemma 2 and provides that p_j is a hyperbolic repeller fixed point for j odd and a hyperbolic attractor fixed point for j even. In this way, we conclude that the orbits γ_j^1, passing through p_j of Z_ε^1 are hyperbolic repeller limit cycles if j is odd and a hyperbolic attractor limit cycles if j is even.

In an analogous way, considering the second perturbation Z_ε^2, the fixed points of $\varphi_{Z_\varepsilon^2}$, are given by the zeros of F_ε^2. Therefore, by Lemma 3 we get that periodic orbits of Z_ε^2 are given by the trajectories γ_j^2 passing through the points $q_j = (0,0,\varepsilon^2/j), j \in \{2,3,\ldots\}$. As the previous case, the stability is given by the Lemma 3. Moreover, q_j is a hyperbolic repeller fixed point for j odd and a hyperbolic attractor fixed point for j even. Therefore, γ_j^2 are hyperbolic repeller limit cycles if j is odd and a hyperbolic attractor limit cycles if j is even. $\square$

Remark 3 (Asymptotically stability of limit cycles)**.** *One should observe that the results of the current section also guarantee the asymptotical stability of some limit cycles. We remember that these limit cycles are obtained through the perturbation of the center contained in S, stated in Proposition A. Indeed, Proposition 2 assures that, by using Lemmas 4 and 5, the limit cycles coming from the perturbation of such a center are hyperbolic. Moreover, if j is even, they attract every trajectories which are sufficiently close to them, that is, those limit cycles are asymptotical stable.*

5. Proof of the main results of the paper

5.1. Proof of Theorem A and Corollary B.

Proof of Theorem A:. Item (a): It follows from Remark 2.

Item (b): It follows from Proposition 2.

Item (c): It follows from Proposition B and a remark inside the proof of Proposition 2. $\square$

Proof of Corollary B:. Suppose that the codimension of the origin of Z_0, given by (1) is $m < \infty$. Then, in a neighborhood of Z_0 there are PSVFs of m distinct topological types. This is a contradiction due Theorem A. So, the codimension of this singularity is infinite. $\square$

5.2. **Proof of Theorem C.** Let $K \subset \mathbb{R}^3$ be a compact set around the origin. Consider the restrictions to K of Z_0 and of its perturbations. Since it will not cause confusion, we denote $Z_0|_K = Z_0$ and $Z_\varepsilon|_K = Z_\varepsilon$.

Let Z_ε^3 be the perturbation of Z_0 given by (12). By Lemma 6 we get that the first return map is given by

$$\varphi_{Z_\varepsilon^3}(0,0,z) = z - \varepsilon(z - \varepsilon)^{k+1}.$$

We define the application

$$F_\varepsilon^{3,j}(z) = \varepsilon(z - \varepsilon)^j \left(z - \frac{\varepsilon}{2} \right) \ldots \left(z - \frac{\varepsilon}{k-j} \right).$$

Consider $\mathcal{V}$ a neighborhood of Z_0 and $\varepsilon > 0$ sufficiently small such that $Z_\varepsilon^{3,j} \in \mathcal{V}$, where $Z_\varepsilon^{3,j}(x,y,z) = X_{i,\varepsilon}^{3,j}(x,y,z)$, if $(x,y,z) \in R_i$, $i = 1, 2, \ldots, 8$, with

$$(17) \qquad\qquad X_{2,\varepsilon}^{3,j} = X_2 + \left(0, 2\frac{\partial F_\varepsilon^{3,j}}{\partial z}(z), 0 \right),$$

and $X_{i,\varepsilon}^{3,j}(x,y,z) = X_i(x,y,z)$ for $i = 1, 3, 4, 5, 6, 7, 8$ are given by (2). Associated to the family (17) we have the normalized sliding vector fields given by

$$Z_{1,4,\varepsilon}^{3,j,s}(x,y,z) = Z_{1,4}^s(x,y,z) = (1, 0, -1)$$

$$Z_{1,5,\varepsilon}^{3,j,s}(x,y,z) = Z_{1,5}^s(x,y,z) = (-1, 1, 0)$$

$$Z_{1,2,\varepsilon}^{3,j,s}(x,y,z) = \left(0, -1 + \frac{\partial F_\varepsilon^{3,j}}{\partial z}(z), 1 \right).$$

A straightforward calculation shows that the trajectories $\phi_{Z_{1,2,\varepsilon}^{3,j,s}}$ of $Z_{1,2,\varepsilon}^{3,j}$ are parameterized by

$$(18) \qquad\qquad \phi_{Z_{1,2,\varepsilon}^{3,j,s}}(t,p) = (x, -t + F_\varepsilon^{3,j}(t) + y, t + z),$$

where $p = (x, y, z)$.

The next lemma provides the expression of the first return map for $Z_\varepsilon^{3,j}$

Lemma 7. *The first return map of $Z_\varepsilon^{3,j}$ is given by*

$$\varphi_{Z_\varepsilon^{3,j}}(0,0,z) = z - F_\varepsilon^{3,j}(z).$$

Proof. The proof is analogous to that one presented in Lemma 6 and will be omitted. $\square$

Proof of Theorem C. Consider in Ω^r the C^r-topology with $r \geq k + 1$. By Lemma 7, the expression of the first return map of $Z_\varepsilon^{3,j}$ is given by

$$\varphi_{Z_\varepsilon^{3,j}}(0,0,z) = z - F_\varepsilon^{3,j}(z)$$

and therefore the limit cycles are given by the solution of

$$F_\varepsilon^{3,j}(z) = 0.$$

So we conclude that $z_1 = \varepsilon$ is a zero of multiplicity j and $z_2 = \varepsilon/2, \ldots, z_{k-j} = \varepsilon/(k-j)$ are simple zeros of this equation, where $j = 0, 1, \ldots, k - 1$. In this way, we obtain that for each j the perturbation $Z_\varepsilon^{3,j}$ presenting exactly $k - j$ limit cycles.

From Proposition 21 of [8], two PSVFs are equivalent if and only if the first return maps are conjugated. It is easy to see that the first return maps are conjugated if and only if the number of zeros are equal and with the same multiplicity ($k - j$ simple zeros and one zero

of multiplicity j). So, in a neighborhood (in the topology C^{k+1}) of $Z_\varepsilon^{3,j}$ we get exactly k distinctly topologically types of PSVFs, when $j = 0, 1, \ldots, k - 1$.

Therefore, Z_ε^3 has codimension k. $\hfill\square$

6. Conclusion

In this paper we consider a 3D model, given by (1), where eight smooth vector fields are combined, in the Filippov's sense, in order to create a *center* with a triangular shape at the sliding region S, where the *sliding vector field* is defined. Moreover, S is a global attractor for the trajectories of (1).

An important remark is that all smooth vector fields considered, and also the sliding vector fields, are **constant** and even in this case the PSVF Z_0 given by (1) has infinite codimension. This fact reveals the complexity involved in the analysis of piecewise smooth dynamical systems.

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